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A Lagrange representation of cellular automaton traffic-flow models

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Abstract
A new cellular automaton (CA) model of traffic flow in the Lagrange form is proposed in this paper. We study the algebraic relationship between models with the Lagrange form and the Euler form of Burger's CA, which is constructed from Burger's equation using the ultradiscrete method. It is found that the Lagrange form has made the description of traffic flow in one lane simpler. Thus we have extended a simple Lagrange model to include the effects of inertia of cars and drivers' perspective. The extended model shows metastable states and complex phase transition from a free to congested state, which is similar to the observed data for expressways.

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1. Introduction

Recently, the cellular automaton (CA) has been used as a method of modelling complex phenomena in various fields such as fluid dynamics [1] and simple granular dynamics [2]. In this paper, we consider CA modelling for traffic flow, which has been a progressing field over recent years.

Traffic models can be classified into two categories in general: macroscopic and microscopic. For macroscopic models, Burger's equation is used to study shock waves of traffic flow in expressways [3]. Kühne [4] considered a hydrodynamic equation with viscosity, which is similar to Burger's equation and is an extension of the Payne model [5]. Kerner and Konhäuser [6] also proposed an extended model of Burger's equation, which is similar to the Navier–Stokes equations. There are car-following models and CA models in the microscopic model. Among car-following models, the optimal velocity (OV) model have been extensively studied recently [7], and unstable uniform traffic flow can well be described by this model.

CA models are quite simple and flexible, and suitable for computer simulations. Rule 184, one of the elementary CA rules investigated by Wolfram [9], is a prototype of all CA models for traffic flow. Nagel and Schreckenberg (NS) proposed a stochastic and higher speed generalization of rule 184 [8], in which they reproduce the spontaneous formation of jams by introducing braking probability in its evolutional rule. Fukui and Ishibashi (FI) extended rule 184 to the case that the maximum speed of cars is larger than 1, and discussed the effect of the car speed on the phase transition from free to congested flow [10]. The FI model is a deterministic model and considered as a variant of the NS model with different acceleration and breaking rules. Takayasu and Takayasu [11] considered an effect of inertia of cars by introducing the slow-start rule. They successfully showed a simple metastable state around a critical car density of the phase transition, which is considered to play an important role in the formation of jams [12].

In our previous papers, we have revealed a fundamental relationship between macroscopic and microscopic models of traffic flow [13], i.e., we have derived a corresponding CA from Burger's equation using the ultradiscrete method [14]. It is shown that Burger's CA (BCA) becomes rule 184 if we restrict the value set as $\{0, 1\}$. We also extend BCA to a multi-velocity case, and obtain more realistic models for traffic flow [15, 16]. Extending to multi-velocity, however, makes CA models much complicated in their expression since the neighbourhood number becomes large. We have only extended BCA to the case of a maximum velocity of 2 in our previous papers.

The relationship between car-following models and BCA has not been revealed up to now. Moreover, the very different approaches given above in modelling traffic flow and their relation are poorly understood. We have found that it is quite important to introduce the concept of 'Euler' and 'Lagrange' form in traffic models in order to clarify their relation. This nomenclature comes from hydrodynamics and we use the hydrodynamic analogy in this paper. Car-following models are represented by the Lagrange form, while BCA is written in the Euler form. The macroscopic traffic models are all represented by hydrodynamic equations, which are the Euler form. In the Euler form, since the coordinate on a road is considered to be the field variable, each car cannot be traced during the time evolution. In the Lagrange form, each car is considered as a interacting particle and the dependent variable is the position of cars, thus it is easy to trace each car in this model. In the case of small densities of cars, the Lagrange model becomes more efficient for computer simulations [17].

In this paper, we propose some deterministic traffic models using the Lagrange form, and consider the above problems in detail. In section 2, we review BCA as a traffic model of the Euler form, and the relationship between the Euler and Lagrange models is discussed in section 3. We give a new model in the Lagrange form in section 4 and study phase-transition phenomena of this model. Concluding discussions are given in section 5.

2. The Euler form of a traffic model

A road is considered as a field and each car is not distinguished in the Euler form. Thus the Euler form has merits when modelling overtaking the car in front or dealing with the traffic flow of more than one lane.

The models in our previous papers are all represented by the Euler form which is based on BCA. BCA is given by Burger's equation [3] $v_t = 2vv_x + v_{xx}$ through the transformation $U_j^t = L/2 + \varepsilon\Delta x v(j\Delta x, t\Delta t)$ as [13]

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, L - U_j^t) - \min(U_j^t, L - U_{j+1}^t) \tag{1}$$

where Δx and Δt are lattice intervals in x and t, respectively and ε is a parameter used in the ultradiscrete formula [14]. Assuming that $L > 0$ and $0 \leqslant U_j^t \leqslant L$ for any site j at a certain time t, then $0 \leqslant U_j^{t+1} \leqslant L$ holds for any j. Thus (1) is equivalent to a CA with a value set $\{0, 1, \ldots, L\}$. Moreover, if we put a restriction $L = 1$ on BCA, then BCA is equivalent to rule

184. There is a parameter L in this model, and we consider that a site could either represent a longer segment of the expressway capable of accommodating a maximum of L cars, or the unit segment of a L-lane expressway.

The maximum velocity of cars in BCA is 1, and we have extended BCA for the case of maximum velocity 2 [15, 16]. It is shown that the phase-transition behaviour in the extended BCA (EBCA) model agrees well with observed data. Extension of BCA to general velocities is, however, found to be difficult as long as we use the Euler form. Euler models become complex in general when the number of neighbouring sites in an evolutional rule becomes large. If we consider L as the number of the lane [18], the Euler form is good for modelling traffic flow since multi-lane models are difficult to represent in Lagrange form. It is better to use the Lagrange form when we consider one lane phenomena.

3. Lagrange models for traffic flow

In this section, we propose Lagrange traffic-flow models. We can follow the position of each car individually in the Lagrange form and it is suitable for representing the case where the order of cars does not change, i.e., no car overtakes the car in front in the same lane.

In our previous papers, we have studied essential factors that affect the phase transition from the free to congested state, and found that three factors, i.e., perspective of drivers, the maximum velocity and inertia of cars, are quite important in determining the critical density. Driver's perspective means that drivers are assumed to be able to see far away enough to follow several cars in front. In this section, we consider a Lagrange model that includes arbitrary maximum velocity and the driver's perspective.

The basic and prototype model is rule 184, and a Lagrange representation of the rule is given by

$$x_j^{t+1} = x_j^t + \min(1, x_{j+1}^t - x_j^t - 1) \tag{2}$$

where x_j^t is the position of the jth car from the leftmost one at time t, and the second term in the right-hand side (rhs) represents the velocity of the jth car. $x_{j+1}^t - x_j^t - 1 \equiv h_j^t$ is the headway and this represents the number of vacant sites between the jth and $(j+1)$th car. In the following, we assume that a road is a single lane and cars move to the right. From (2), we see that a car can move forward one site only when their front site is empty, and this is nothing but rule 184. Let us extend (2) to the case that the maximum velocity of cars is V. In this case, cars can move forward at most V sites per unit time if their front sites are empty enough. This is easily modelled by

$$x_j^{t+1} = x_j^t + \min(V, h_j^t). \tag{3}$$

This corresponds to the Lagrange form of the FI model. In (3), a car follows only the car in front. Let us extend to the case that the number of cars that a driver can see in front is S. The model is given by

$$\begin{aligned} x_j^{t+1} &= x_j^t + \min(V, x_{j+S}^t - x_j^t - S) \\ &= x_j^t + \min(V, h_j^t + \cdots + h_{j+S-1}^t). \end{aligned} \tag{4}$$

Here let us consider a relationship between these Lagrange models and BCA in the Euler form. First, we derive a general equation that relates the Lagrange variable x or h and the Euler variable U, which holds irrespective of the evolution rule of CA. U can be generally represented using the headway h when the configuration of cars is given at time t. When we consider variables at the same value of t, we omit the subscript t and write $U_{x_i^t}^t = U(x_i)$, for example. Then relations

$$U(x_i) = 1 \tag{5}$$

$$U(x_i + 1) = \delta(h_i) \tag{6}$$

apparently hold and here $\delta(x)$ is a delta function, where $\delta(x) = 0$ everywhere except $\delta(0) = 1$. It can be written using the max function as

$$\delta(x) = \max(0, \min(1 + x, 1 - x)) \tag{7}$$

when x is an integer. Equation (6) indicates that the front site of the ith car is occupied only when its headway is zero. Whether the site $x_i + k$ $(k \geqslant 1)$ is occupied or not is generally determined by $h_i, \ldots, h_{i+k-1}$. Considering the configuration from the ith to $(i + k)$th car, we obtain the relation

$$U(x_i + k) = \sum_{l=0}^{k-1} \delta(h_i + \cdots + h_{i+l} - (k - l - 1)) \tag{8}$$

for $k \geqslant 1$, which connects the Euler variable and Lagrange variable. Moreover, we can derive two other relations that hold irrespective of the evolutional rules. First, if the ith car does not move at time t, then $x_i^{t+1} = x_i^t$ holds. Then we obtain

$$U^{t+1}_{x_i^{t+1}} - U^{t+1}_{x_i^t} = \delta(x_i^{t+1} - x_i^t). \tag{9}$$

The second one is about the Euler–Lagrange transformation and comes from the continuous theory. Let us consider the equation

$$U^{t+1}_{x_j^{t+1}} - U^t_{x_j^t} = 0. \tag{10}$$

This is trivial because the number of cars does not change in the time evolution, and (10) is identical to $1 - 1 = 0$ because of (5). If we write the Lagrange derivative as D, then (10) becomes $\mathrm{D}U/\mathrm{D}t = 0$ for a continuous limit. Equation (10) is equivalently written as

$$U^{t+1}_{x_j^{t+1}} - U^{t+1}_{x_j^t} + U^{t+1}_{x_j^t} - U^t_{x_j^t} = 0. \tag{11}$$

It is easily seen that (11) becomes $\mathrm{d}x/\mathrm{d}t \cdot \partial U/\partial x + \partial U/\partial t = 0$ for a continuous limit, which is the usual form of the Euler derivative. Since formulae (8)–(10) do not depend on the evolutional rules of CA, they are quite important in the Euler–Lagrange transformation. It should be noted that these formulae hold in general rules that satisfy the exclusion principle, i.e., a car cannot enter a site which is already occupied by another car.

Let us study the correspondence between (1) with $L = 1$ and (2) using these formulae. Substituting $j = x_i^t$ into (1) with $L = 1$, we obtain

$$U^{t+1}_{x_i^t} = U^t_{x_i^t} + \min(U^t_{x_i^t-1}, 1 - U^t_{x_i^t}) - \min(U^t_{x_i^t}, 1 - U^t_{x_i^t+1}). \tag{12}$$

Using the relation $U^t_{x_i^t} = 1$, we can simplify (12) to

$$U^{t+1}_{x_i^t} = U^t_{x_i^t+1}. \tag{13}$$

From (11) and (13), we get

$$U^{t+1}_{x_j^{t+1}} - U^{t+1}_{x_j^t} = 1 - U^t_{x_j^t+1}. \tag{14}$$

Substituting (6) and (9) into (14) and taking $h_i^t \geqslant 0$ into account, we finally obtain

$$1 - \delta(x_i^{t+1} - x_i^t) = 1 - \max(0, 1 - h_i^t) = \min(1, h_j^t). \tag{15}$$

In the case of rule 184, since the maximum velocity is 1, the left-hand side of (15) can equivalently be rewritten as $x_i^{t+1} - x_i^t$, thus (15) becomes (2).

Next, we consider the case $L \geqslant 2$ in (1). Since we are considering the Lagrange form, the road is a single lane and the site represents a segment capable of accommodating a maximum of L cars. For simplicity, we first consider the case $L = 2$. In (1) with $L = 2$, we divide each site into two subsites and each subsite is assumed to contain at most one car. Then from (1) the dynamics for subsites in the Euler form is given by

$$\begin{aligned} C_j^{t+1} = C_j^t &- \min(C_j^t, \max(1 - C_{j+1}^t, 1 - C_{j+2}^t)) \\ &+ \min(C_{j-2}^t, 1 - \text{majority}(C_{j-1}^t, C_j^t, C_{j+1}^t)) \\ &+ \min(C_{j-1}^t, C_{j+2}^t, \text{xor}(C_j^t, C_{j+1}^t)) \\ &\equiv C_j^t + W_j^t \end{aligned} \tag{16}$$

where majority(A, B, C) is the majority function and xor(A, B) is the exclusive OR (XOR) function [19] defined in canonical form as

$$\begin{aligned} \text{majority}(A, B, C) = \max(&\min(1 - A, B, C), \min(A, 1 - B, C), \\ &\min(A, B, 1 - C), \min(A, B, C)) \end{aligned} \tag{17}$$

$$\text{xor}(A, B) = \max(\min(1 - A, B), \min(A, 1 - B)). \tag{18}$$

For example, majority$(0, 1, 1) = 1$ because the number of 1 is larger than 0. In (16), the term $\min(C_j^t, \max(1 - C_{j+1}^t, 1 - C_{j+2}^t))$ shows that the car in site j can move forward if site $j + 1$ or $j + 2$ is empty. The term including the majority function represents the number of cars at site $j - 2$ entering site j with maximum velocity 2, and the term including the XOR function is the number of cars at site $j - 1$ entering site j with velocity 1. It can easily be verified that (16) is equivalent to (1) with $L = 2$. Let us define the variable transformation as

$$U_{j+k}^t = C_{j+2k}^t + C_{j+2k+1}^t \tag{19}$$

for any integer k, then we obtain from (16)

$$U_j^{t+1} = U_j^t + W_j^t + W_{j+1}^t. \tag{20}$$

Substituting (19) into (1) to obtain

$$U_j^{t+1} = U_j^t + \min(C_{j-2}^t + C_{j-1}^t, 2 - C_j^t - C_{j+1}^t) - \min(C_j^t + C_{j+1}^t, 2 - C_{j+2}^t - C_{j+3}^t). \tag{21}$$

The rhs of (20) and (21) are both functions of six variables from C_{j-2}^t to C_{j+3}^t. Each variable takes the value 0 or 1, then there are $2^6 = 64$ cases in total. Comparing values of the rhs of (20) and (21) in the 64 cases numerically, it is found that in all cases they show the same value. Thus the two expressions are shown to be equivalent.

Since (16) satisfies the exclusive principle, we can write a Lagrange form equivalent to (16) as

$$x_j^{t+1} = x_j^t + \min(2, h_j^t + h_{j+1}^t). \tag{22}$$

We have obtained this expression physically by considering the motion of cars in (16), and found that the description becomes much simpler. Using similar arguments to those given above, the Lagrange form for BCA with general L is given by

$$x_j^{t+1} = x_j^t + \min(L, h_j^t + \cdots + h_{j+L-1}^t). \tag{23}$$

These Lagrange forms are obtained from 'physical intuition'. The exact Euler–Lagrange transformation from (1) to (23) is unknown and we have only studied the algebraic relation for the case of $L = 1$.

Comparing (23) and (4), it is found that BCA is equivalent to the model in which both the maximum velocity and the number of viewable cars in front are equal to L. Moreover, an interesting property of BCA is revealed from the consideration of (20) and (21). If a partition

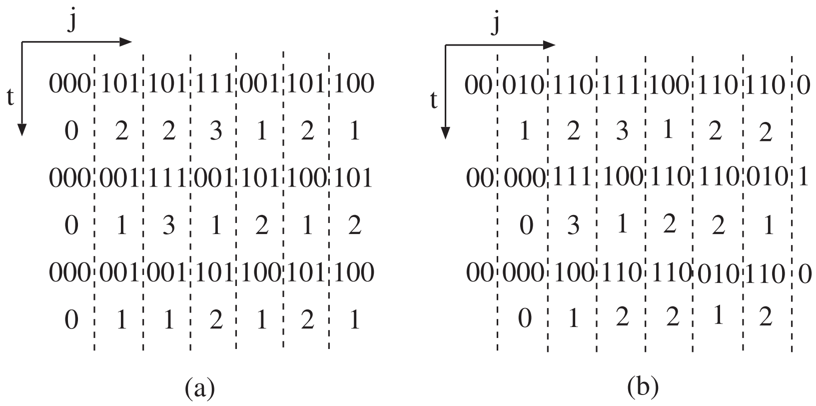

Figure 1. Correspondence between the Lagrange model (23) and BCA (1) for $L = 3$. Each car moves according to (23) and we put a partition at every three sites on the road. (*a*) and (*b*) are the same time evolution but the position of the partition is different. It is seen that in both cases the sum of the cars between partitions obeys BCA with $L = 3$.

is put at every L sites and cars move on the road according to (23), then the sum of the number of cars between two successive partitions obeys BCA even if the position of the partition is shifted (figure 1).

Finally, we comment on a relationship between the OV model and the FI model. Equation (3) is rewritten as

$$x_j^{t+1} - 2x_j^t + x_{j-1}^t = \min(V, h_j^t) - (x_j^t - x_j^{t-1}). \tag{24}$$

Taking a continuous limit $\Delta t \to 0$ and after appropriate scaling, we obtain

$$\frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} = \min(V, h_j^t) - \frac{\mathrm{d}x_j}{\mathrm{d}t}. \tag{25}$$

This is nothing but the OV model, and $\min(V, h_j^t)$ is the same as the piecewise linear OV function in [20].

4. An extended Lagrange model

In this section, let us extend our Lagrange model in order to include the effects of the inertia of cars. The inertia effect can be represented by the slow-start rule. The velocity of cars depends not only on the present headway but also on the past headway in the rule [16]. Thus the Lagrange form of (2) with the slow-start rule is given by

$$x_j^{t+1} = x_j^t + \min(1, x_{j+1}^t - x_j^t - 1, x_{j+1}^{t-1} - x_j^{t-1} - 1). \tag{26}$$

This is a Lagrange representation of the model proposed in [11]. Considering the continuous limit discussed in the last part of the previous section, it is found that the OV function in the slow-start rule depends not only on the present headway but also on the differential of the headway with respect to time. This is a new property and is not seen in the original OV model.

Next, combining (26) and (4), we finally obtain the Lagrange model which includes the three factors as

$$\begin{aligned} x_j^{t+1} &= x_j^t + \min\left(V_j^t, \min_{k=1,\ldots,S-1}\left(\sum_{i=0}^{k} h_{j+i}^t + V_{j+k}^t\right)\right) \\ &= x_j^t + \min\left(V_j^t, \min_{k=1,\ldots,S-1}\left(x_{j+k}^t - x_j^t - k + x_{j+S+k}^{t-1} - x_{j+k}^{t-1} - S\right)\right) \end{aligned} \tag{27}$$

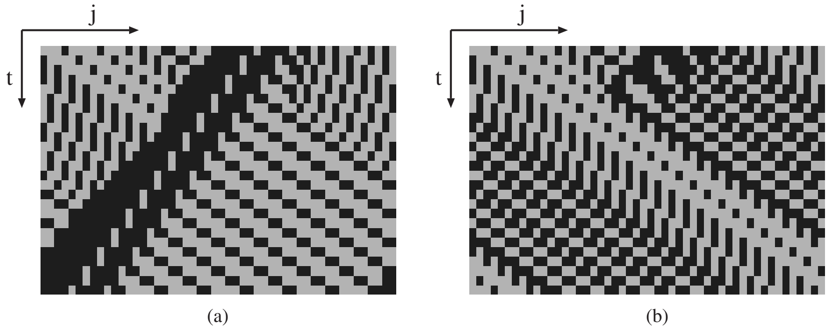

Figure 2. Typical time evolution for the case of $V = 2$ and $\rho = 0.48$ under the periodic boundary condition. (a) is the case $S = 2$ and (b) $S = 4$. The black and grey squares represent 1 and 0, respectively. We see that the congested state disappears as the perspective parameter S becomes larger.

where

$$V_j^t = \min\left(V, \sum_{i=0}^{S-1} h_{j+i}^t, \sum_{i=0}^{S-1} h_{j+i}^{t-1}\right)$$
$$= \min\left(V, x_{j+S}^t - x_j^t - S, x_{j+S}^{t-1} - x_j^{t-1} - S\right). \tag{28}$$

The term $\min_{k=1,\dots,S-1}(\cdots)$ in the rhs of (27) plays an important role in avoiding cars colliding. This is because the condition that a car does not overtake all the viewable cars in its front is given by

$$\sum_{i=0}^{k} h_{j+i}^t + V_{j+k}^t \geqslant V_j^t \qquad k = 1, \dots, S-1. \tag{29}$$

Equation (27) is also written in the conservation form using the headway as

$$h_j^{t+1} = h_j^t + \min\left(V_{j+1}^t, \min_{k=1,\dots,S-1}\left(\sum_{i=0}^{k-1} h_{j+i+1}^t + \sum_{i=k}^{S+k-1} h_{j+i+1}^{t-1}\right)\right)$$
$$- \min\left(V_j^t, \min_{k=1,\dots,S-1}\left(\sum_{i=0}^{k-1} h_{j+i}^t + \sum_{i=k}^{S+k-1} h_{j+i}^{t-1}\right)\right). \tag{30}$$

Next, we investigate the behaviour of cars in this new Lagrange model. Typical time evolutions under the periodic boundary condition are given in figure 2. The maximum velocity is fixed at $V = 2$, and the number of viewable cars is changed from $S = 2$ (figure 2(a)) to $S = 4$ (figure 2(b)). The initial car density is set at $\rho = 0.48$ in both cases, where car density is defined as the ratio of the total number of cars to the total number of sites in a period. We see that a congested state propagates backward in $S = 2$, while the congested states will disperse when $S = 4$.

The fundamental diagram, which is the plot of the traffic flow Q and density ρ, is given by figure 3. The flow Q is defined by the multiplication of the density and the average velocity of cars. From figure 3, we observe a complex phase transition from a free to a congested state around the critical density. There are many metastable branches in the diagram, as seen in our previous models [16]. Particularly, we see two small metastable branches between the densities of 0.2 and 0.4. It is seen that the phase transition from free to congested states is discontinuous (first order). Moreover, the sample points are spread over a two-dimensional

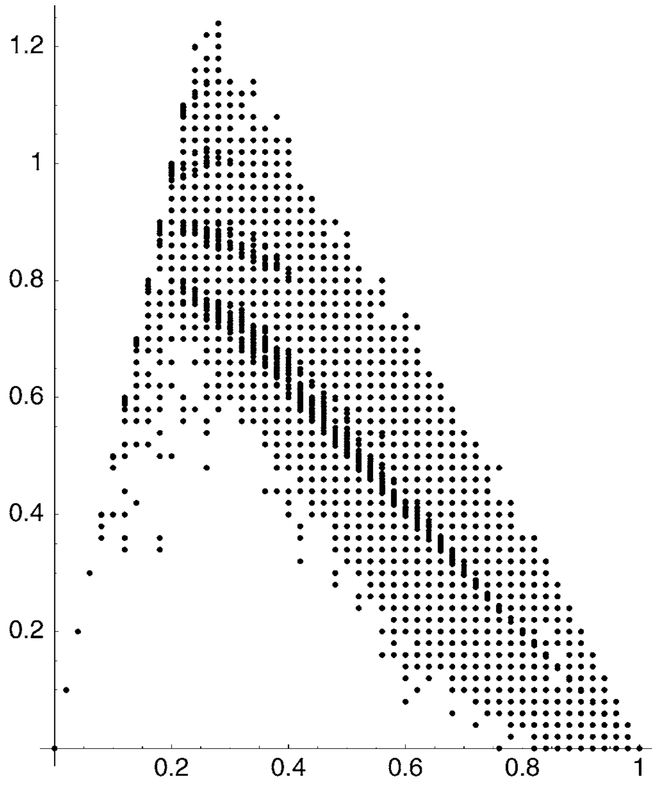

Figure 3. A fundamental diagram of the new Lagrange model. The parameters are set at $V = 5$ and $S = 2$ and the spatial period is 50 sites. The initial car density is varied from 0 to 1 with step size 0.05. At each density, we start calculations from 25 randomly generated initial configurations, and superpose all plots in the diagram from time 0 to 100. The average flows over the 25 cases for each time value are plotted by full circles. We see various metastable branches in the diagram.

area even if we average out over many initial conditions. This implies that the average velocity is non-stationary around the critical density. The wide scattering is surprising since our model is completely deterministic. The schematic diagram is given by figure 4. For the state A, the stationary pattern

$$\underbrace{11\cdots11}_{S}\underbrace{000\cdots000}_{V} \tag{31}$$

is observed and so we can see that the density and average velocity of the state are $S/(V+S)$ and V, respectively. Also at B, from the stationary pattern

$$\underbrace{11\cdots11}_{S}\underbrace{000\cdots000}_{2V} \tag{32}$$

the density and velocity become $S/(2V+S)$ and V, respectively. Thus it is found that the position of the maximum current state A is

$$\left(\frac{S}{V+S}, \frac{SV}{V+S}\right) \tag{33}$$

and the branching point B is

$$\left(\frac{S}{2V+S}, \frac{SV}{2V+S}\right). \tag{34}$$

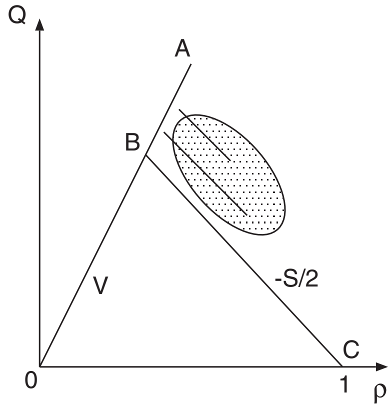

Figure 4. A schematic fundamental diagram of the new model. The line O–A represents the free flow with gradient V. The line B–C shows the strong jam with gradient $-S/2$. The dotted region represents the weak jam, in which the flow is relatively high in the congested state. We see that there are many metastable branches in it.

The branch A–B represents overdense free flow and is stable under weak perturbation. It will be a congested state due to strong perturbation [15]. In figure 4, there are other metastable branches in the dotted region of the congested state. These complex metastable branches resemble those of the EBCA model in the Euler form proposed in [16]. It should be noted that this high-current congested state is similar to the synchronized state proposed by Kerner and Rehborn [21]. However, our model is deterministic and deals with one lane without bottlenecks. The synchronized state typically occurs from downstream the bottleneck in more than one lane. Also, cross-correlation between the density and flow vanishes in the state, which leads to wide scattering of the data [22]. Thus, our findings are different from the synchronized state from a physical point of view. Synchronized flow can also be found on one-lane roads, but in this case the phase transition is not considered to be of first order and is no longer hysteretic in nature [12]. In our model, there is the first-order phase transition from free to congested states. Also, there exist both metastable ‘weak jam’ (in the scattering region) and the ‘strong jam’ (line B–C) in the congested state. Thus the transition of ‘free $\rightarrow$ weak jam $\rightarrow$ strong jam’ is generally observed in our model.

5. Discussions

In this paper, we consider traffic CA models in the Lagrange form, which is simple compared to the Euler form in the case that the order of cars on a road does not change or considering just one lane. A new Lagrange model, including drivers’ perspective, the arbitrary maximum velocity and inertia of cars, is proposed and shows complex metastable states which are seen in the observed data.

We also discuss the relation between the Euler and Lagrange form. To date this has received less attention than the relationship between the two CA forms. We think that this study is quite important in constructing a general theory of traffic models ranging from microscopic to macroscopic ones. The relationship between rule 184 in the Lagrange form and BCA with $L = 1$ becomes clear, and the general expression of BCA in the Lagrange form is obtained in this paper. There are, however, some problems in the Euler–Lagrange transformation. In continuous equations such as the Euler equation in fluid dynamics, there is a clear transformation between the two forms, i.e., a kind of Taylor expansion connects the two expressions. In CA, the general framework of such a direct transformation is unknown. We believe that the formulae (8)–(10) have an important role in transforming the two forms when the model satisfies the exclusion principle. Of course not only these formulae but also other algebraic methods are needed to investigate further the relationship between the two models. We are now investigating the relation between the EBCA model, which is represented in the Euler form, and the new Lagrange model proposed in this paper. Fundamental diagrams of both models are qualitatively similar, especially respecting the metastable branches of the congested state which exist in both models. Models which show such metastable states are, as far as we know, the only two cases of all the traffic models. Since the wide scattering of data in the congested state around the critical density is typically observed in real expressway traffic [23], it is important to analyse further the complex phase transition of our models and physically clarify the scattering region. Creating the best traffic model which combines the good properties of both the Euler and Lagrange models is an important future task.

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